THE COEFFICIENT COALGEBRA OF A SYMMETRIZED TENSOR SPACE

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Abstract. The coefficient coalgebra of r-fold tensor space and its dual, the Schur algebra, are generalized in such a way that the role of the symmetric group Σ_r is played by an arbitrary subgroup of Σ_r. The dimension of the coefficient coalgebra of a symmetrized tensor space is computed and the dual of this coalgebra is shown to be isomorphic to the analog of the Schur algebra.

0. Introduction

Let K be the field of complex numbers. The vector space $E = K^n$ is naturally viewed as a (left) module for the group algebra $K\Gamma$ of the general linear group $\Gamma = \mathrm{GL}_n(K)$. The r-fold tensor product $E^{\otimes r}$ is in turn a module for $K\Gamma^r$, where $\Gamma^r = \Gamma \times \cdots \times \Gamma$ (r factors). Let G be a subgroup of the symmetric group Σ_r and let $\chi : G \to K$ be an irreducible character of G. The "symmetrized tensor space" associated with χ is $E^\chi = E^{\otimes r} t_\chi$, where t_χ is the central idempotent of the group algebra KG corresponding to χ (with the action of G on $E^{\otimes r}$ being given by place permutation).

In this paper, we study the coefficient coalgebra A_χ of the R-module E^χ, where R is the subalgebra of $K\Gamma^r$ consisting of those elements fixed by G under the action of Σ_r on Γ^r given by entry permutation. (If V is an R-module with finite K-basis $\{v_j \mid j \in J\}$, then for each $\kappa \in R$, we have $\kappa v_j = \sum_i \alpha_{ij}(\kappa) v_i$ for some $\alpha_{ij} : R \to K$. The linear span of the functions α_{ij} ($i, j \in J$) has a natural coalgebra structure. It is the "coefficient coalgebra" of V, denoted $\mathrm{cf}(V)$.)

Section 1 sets up notation and presents some standard results suitably generalized to the current situation.

In Section 2, we obtain generalizations of some classical results (i.e., results for the case $G = \Sigma_r$). In particular, we show that the image S of the representation afforded by the R-module $E^{\otimes r}$ equals the set of those endomorphisms of $E^{\otimes r}$ that commute with the action of G (Theorem 2.7). This generalizes Schur's Commutation Theorem [Ma, 2.1.3] in the classical case, where S is the "Schur algebra." We also observe that the algebra isomorphism $S \cong A^*$ ($A = \mathrm{cf}(E^{\otimes r})$) in the classical case [Ma, 2.3.5 and following paragraph] continues to hold with G arbitrary (Theorem 2.8).

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In Section 3, we study the coefficient coalgebra A_χ and the corresponding analog S_χ of the Schur algebra and establish an algebra isomorphism $S_\chi \cong A_\chi^*$ (Theorem 3.4). We exhibit decompositions of A and S in terms of the various A_χ and S_χ, respectively (Theorems 3.3 and 3.5), and end by providing a formula for the dimension of A_χ over K (Theorem 3.6).

Some of the results of this paper appear in the Ph.D. dissertation [Tu] of the second author written under the direction of the first author. We thank the referee for some useful suggestions.

1. Notation and background

Let G be a fixed subgroup of the symmetric group Σ_r.

For the general results of this section and the next, the field K can be more general than the field of complex numbers. We assume only that K is an infinite field of characteristic not a divisor of $|G|$. Fix a positive integer n. The vector space $E = K^n$ is acted on naturally (from the left) by the semigroup $M = \mathrm{Mat}_n(K)$ of $n \times n$ matrices over K and is therefore a KM-module, where KM is the semigroup algebra of M over K.

The group G acts from the right on $M^r = M \times \cdots \times M$ (r factors) by $m\sigma = (m_{\sigma(1)}, m_{\sigma(2)}, \ldots, m_{\sigma(r)})$ $(m \in M^r, \sigma \in G)$.

The semigroup M^r acts on $E^{\otimes r} = E \otimes \cdots \otimes E$ (r factors) by $mv = m_1 v_1 \otimes \cdots \otimes m_r v_r$ $(m \in M^r, v = v_1 \otimes \cdots \otimes v_r \in E^{\otimes r})$. This action extends linearly to make $E^{\otimes r}$ a KM^r-module.

For each $1 \leq \alpha \leq n$, let $e_\alpha \in E$ be the n-tuple with β-entry $\delta_{\alpha\beta}$ (Kronecker delta). Let $I = \{i = (i_1, \ldots, i_r) \in \mathbf{Z}^r \mid 1 \leq i_j \leq n \text{ for all } j\}$. The space $E^{\otimes r}$ has basis $\{e_i \mid i \in I\}$, where $e_i := e_{i_1} \otimes \cdots \otimes e_{i_r}$.

For $m \in M^r$ and $i, j \in I$, define $m_{i,j} = \prod_{\alpha=1}^r (m_\alpha)_{i_\alpha j_\alpha}$.

1.1 Lemma. *Let $m \in M^r$.*

(i) $me_j = \sum_i m_{i,j} e_i \quad (j \in I)$.

(ii) $(m\sigma)_{i,j} = m_{i\sigma^{-1}, j\sigma^{-1}} \quad (\sigma \in G,\ i, j \in I)$.

Proof. (i) For all $j \in I$, we have

$$me_j = m_1 e_{j_1} \otimes \cdots \otimes m_r e_{j_r} = \left(\sum_{i_1=1}^n (m_1)_{i_1 j_1} e_{i_1}\right) \otimes \cdots \otimes \left(\sum_{i_r=1}^n (m_r)_{i_r j_r} e_{i_r}\right)$$
$$= \sum_{i \in I} \prod_{\alpha=1}^r (m_\alpha)_{i_\alpha j_\alpha} e_{i_1} \otimes \cdots \otimes e_{i_r} = \sum_{i \in I} m_{i,j} e_i.$$

(ii) For all $\sigma \in G$ and $i, j \in I$, we have

$$\begin{aligned}(m\sigma)_{i,j} &= \prod_{\alpha=1}^r (m_{\sigma(\alpha)})_{i_\alpha j_\alpha} = \prod_\alpha (m_\alpha)_{i_{\sigma^{-1}(\alpha)} j_{\sigma^{-1}(\alpha)}} \\ &= \prod_\alpha (m_\alpha)_{(i\sigma^{-1})_\alpha (j\sigma^{-1})_\alpha} \\ &= m_{i\sigma^{-1}, j\sigma^{-1}},\end{aligned}$$

□

Let $\Gamma = \mathrm{GL}_n(K) \subseteq M$ and put $\Gamma^r = \Gamma \times \cdots \times \Gamma \subseteq M^r$. The right action of G on M^r induces an action on the group algebra $K\Gamma^r$ of Γ^r over K. Denote by R the subalgebra of $K\Gamma^r$ consisting of those elements fixed by G:

$$R = (K\Gamma^r)^G = \{\kappa \in K\Gamma^r \mid \kappa\sigma = \kappa \text{ for all } \sigma \in G\}.$$

The set $\{\bar{g} \mid g \in \Gamma^r\}$ is a K-basis for R, where

$$\bar{g} = \frac{1}{|G|} \sum_{\sigma \in G} g\sigma.$$

Let V be an R-module with finite K-basis $\{v_j \mid j \in J\}$. For $\kappa \in R$ and $j \in J$, we have

$$\kappa v_j = \sum_{i \in J} \alpha_{ij}(\kappa) v_i$$

for some $\alpha_{ij} \in R^* = \mathrm{Hom}_K(R, K)$ (dual space of R). The K-linear span of the set $\{\alpha_{ij} \mid i, j \in J\}$ is the **coefficient space** of the R-module V, denoted $\mathrm{cf}(V)$. This space is independent of the choice of basis for V.

If $R_K(R)$ is the representative K-bialgebra of the multiplicative semigroup R, then V is a right $R_K(R)$-comodule with structure map $\psi : V \to V \otimes R_K(R)$ given by $\psi(v_j) = \sum_{i \in J} v_i \otimes \alpha_{ij}$. We have

$$\triangle(\alpha_{ij}) = \sum_k \alpha_{ik} \otimes \alpha_{kj},$$

$$\epsilon(\alpha_{ij}) = \delta_{ij},$$

so that $\mathrm{cf}(V)$ is a subcoalgebra of $R_K(R)$ [Ab, p. 125].

Let $\rho : R \to \mathrm{End}_K(V)$ be the representation afforded by the R-module V.

1.2 Lemma.

(i) *For $\kappa \in R$, we have $\kappa \in \ker \rho$ if and only if $f(\kappa) = 0$ for all $f \in \mathrm{cf}(V)$.*
(ii) *For $f \in R^*$, we have $f \in \mathrm{cf}(V)$ if and only if $f(\kappa) = 0$ for all $\kappa \in \ker \rho$.*

Proof. See proof of [Ma, 2.2.1]. □

1.3 Theorem. *The map $\psi : \mathrm{im}\,\rho \to \mathrm{cf}(V)^*$ given by $\psi(\rho(\kappa))(c) = c(\kappa)$ is a K-isomorphism.*

Proof. The bilinear map $\mathrm{im}\,\rho \times \mathrm{cf}(V) \to K$ given by $\langle \rho(\kappa), c \rangle = c(\kappa)$ is well defined and nondegenerate by Lemma 1.2. Since the spaces $\mathrm{im}\,\rho$ and $\mathrm{cf}(V)$ are finite dimensional, the induced map $\psi : \mathrm{im}\,\rho \to \mathrm{cf}(V)^*$ is a K-isomorphism. □

2. Generalized coefficient coalgebra and Schur algebra

We continue to assume that K is an infinite field of characteristic not a divisor of $|G|$. Recall that R is the subalgebra of $K\Gamma^r$ consisting of those elements fixed by G.

For $i, j \in I$ define $c_{i,j} : R \to K$ by putting

$$c_{i,j}(\bar{g}) = \frac{1}{|G|} \sum_{\sigma \in G} (g\sigma)_{i,j}$$

($g \in \Gamma^r$) and extending linearly to R.

2.1 Lemma. *The function $c_{i,j}$ is well-defined.*

Proof. Let $g, h \in \Gamma^r$ and assume that $\bar{g} = \bar{h}$. Then $h = g\tau$ for some $\tau \in G$, so

$$c_{i,j}(\bar{h}) = \frac{1}{|G|} \sum_{\sigma \in G} (h\sigma)_{i,j} = \frac{1}{|G|} \sum_{\sigma \in G} (g\tau\sigma)_{i,j} = c_{i,j}(\bar{g}).$$

□

2.2 Lemma. *For $\kappa \in R$ and $j \in I$,*

$$\kappa e_j = \sum_{i \in I} c_{i,j}(\kappa) e_i.$$

Proof. Let $g \in \Gamma^r$. It suffices to establish the equality in the case $\kappa = \bar{g}$. For $j \in I$ we have

$$\bar{g} e_j = \frac{1}{|G|} \sum_{\sigma \in G} (g\sigma) e_j = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{i \in I} (g\sigma)_{i,j} e_i = \sum_{i \in I} c_{i,j}(\bar{g}) e_i,$$

where the second equality uses Lemma 1.1. □

The group G acts on I from the right by $i\sigma = (i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(r)})$. In turn, G acts on $I \times I$ diagonally: $(i, j)\sigma = (i\sigma, j\sigma)$. For $i, j, k, l \in I$, put $(i, j) \sim (k, l)$ if $(k, l) = (i, j)\sigma = (i\sigma, j\sigma)$ for some $\sigma \in G$.

2.3 Lemma. *$c_{i,j} = c_{k,l}$ if and only if $(i, j) \sim (k, l)$.*

Proof. For $i, j \in I$, define $\hat{c}_{i,j} : M^r \to K$ by

$$\hat{c}_{i,j}(m) = \frac{1}{|G|} \sum_{\sigma \in G} (m\sigma)_{i,j},$$

and note that for $g \in \Gamma^r$ we have $\hat{c}_{i,j}(g) = c_{i,j}(\bar{g})$.

Let $i, j, k, l \in I$ and assume that $c_{i,j} = c_{k,l}$. Then $\hat{c}_{i,j}(g) = \hat{c}_{k,l}(g)$ for each $g \in \Gamma^r$ and, since Γ^r is Zariski dense in M^r, it follows that $\hat{c}_{i,j} = \hat{c}_{k,l}$.

Define $b_{i,j} = (b_{i_1 j_1}, \ldots, b_{i_r j_r}) \in M^r$, where $(b_{ab})_{cd} = \delta_{(a,b),(c,d)}$ (Kronecker delta) and note that $(b_{i,j})_{x,y} = \delta_{(i,j),(x,y)}$. For $x, y \in I$, we get, using Lemma

1.1,

$$\begin{aligned}\hat{c}_{x,y}(b_{i,j}) &= \frac{1}{|G|}\sum_{\sigma\in G}(b_{i,j}\sigma)_{x,y}\\ &= \frac{1}{|G|}\sum_{\sigma\in G}(b_{i,j})_{x\sigma^{-1},y\sigma^{-1}}\\ &= \begin{cases}\frac{|G_{(i,j)}|}{|G|}, & (x,y)\sim(i,j)\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

where $G_{(i,j)}$ is the stabilizer of (i,j) in G. Since

$$\hat{c}_{k,l}(b_{i,j}) = \hat{c}_{i,j}(b_{i,j}) = \frac{|G_{(i,j)}|}{|G|} \neq 0,$$

we conclude that $(k,l)\sim(i,j)$.

Now assume that $(i,j)\sim(k,l)$ so that $(k,l)=(i\sigma,j\sigma)$ for some $\sigma\in G$. For any $g\in\Gamma^r$, Lemma 1.1 gives

$$(g\sigma^{-1})_{i,j} = g_{i\sigma,j\sigma} = g_{k,l},$$

so

$$\begin{aligned}c_{i,j}(\bar{g}) &= \frac{1}{|G|}\sum_{\tau\in G}(g\tau)_{i,j} = \frac{1}{|G|}\sum_{\tau\in G}(g\tau\sigma^{-1})_{i,j}\\ &= \frac{1}{|G|}\sum_{\tau\in G}(g\tau)_{k,l} = c_{k,l}(\bar{g}).\end{aligned}$$

Therefore, $c_{i,j}=c_{k,l}$. □

Let A be the coefficient space of the R-module $E^{\otimes r}$.

2.4 THEOREM.

(i) *The space A has K-basis $\{c_{i,j}\,|\,(i,j)\in B\}$, where B is a set of representatives for the orbits of $I\times I$ under the diagonal action of G.*

(ii) *A is a coalgebra with structure maps*

$$\triangle(c_{i,j}) = \sum_{k\in I} c_{i,k}\otimes c_{k,j},$$

$$\epsilon(c_{i,j}) = \delta_{i,j}.$$

Proof. (i) By Lemma 2.2, the space A is the K-linear span of $\{c_{i,j}\,|\,i,j\in I\}$, which equals $\{c_{i,j}\,|\,(i,j)\in B\}$ by Lemma 2.3.

Suppose that $\sum_{(i,j)\in B}\alpha_{i,j}c_{i,j} = 0$ with $\alpha_{i,j}\in K$. Let $\hat{c}_{i,j}: M^r\to K$ be as in the proof of Lemma 2.3. Since $\hat{c}_{i,j}(g)=c_{i,j}(\bar{g})$, we have $\sum_{(i,j)\in B}\alpha_{i,j}\hat{c}_{i,j}(g) = 0$ for all $g\in\Gamma^r$. Since Γ^r is Zariski dense in M^r, we have $\sum_{(i,j)\in B}\alpha_{i,j}\hat{c}_{i,j}=0$. Let $(x,y)\in B$. With $b_{x,y}$ as in the proof of Lemma 2.3 we have (using the formula in that proof)

$$\alpha_{x,y}=\frac{|G|}{|G_{(x,y)}|}\sum_{(i,j)\in B}\alpha_{i,j}\hat{c}_{i,j}(b_{x,y})=0.$$

Therefore, $\{c_{i,j}\,|\,(i,j)\in B\}$ is linearly independent.

(ii) This follows from Lemma 2.2 and the discussion in Section 1. □

The group G acts on $E^{\otimes r}$ by $e_i\sigma=e_{i\sigma}$ $(i\in I)$ and this action extends linearly to make $E^{\otimes r}$ a right KG-module. In fact $E^{\otimes r}$ is a (KM^r,KG)-bimodule. One checks that $v\sigma=v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(r)}$ for all $v=v_1\otimes\cdots\otimes v_r\in E^{\otimes r}$.

The set $N=\mathrm{End}_K(E^{\otimes r})$ is a right KG-module with action given by $(f\sigma)(v)=f(v\sigma^{-1})\sigma$ $(\sigma\in G)$. Moreover, $N^G=\mathrm{End}_{KG}(E^{\otimes r})$, where N^G is the set of those elements of N fixed by G.

Let $\hat{T}:K\Gamma^r\to\mathrm{End}_K(E^{\otimes r})$ be the representation afforded by the $K\Gamma^r$-module $E^{\otimes r}$.

2.5 Lemma. *$\hat{T}$ is a KG-homomorphism.*

Proof. Let $g\in\Gamma^r$, $\sigma\in G$, and $v=v_1\otimes\cdots\otimes v_r\in E^{\otimes r}$. We have

$$\begin{aligned}[\hat{T}(g)\sigma](v)&=\left(\hat{T}(g)(v\sigma^{-1})\right)\sigma\\&=\left(\hat{T}(g)(v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(r)}\right)\sigma\\&=\left(g_1v_{\sigma^{-1}(1)}\otimes\cdots\otimes g_rv_{\sigma^{-1}(r)}\right)\sigma\\&=w_{\sigma(1)}\otimes\cdots\otimes w_{\sigma(r)}\qquad(w_i:=g_iv_{\sigma^{-1}(i)})\\&=g_{\sigma(1)}v_1\otimes\cdots\otimes g_{\sigma(r)}v_r\\&=[\hat{T}(g\sigma)](v).\end{aligned}$$

Therefore, $\hat{T}(g\sigma)=\hat{T}(g)\sigma$ as claimed. □

The K-space N has basis $\{e_{i,j}\,|\,i,j\in I\}$, where

$$e_{i,j}(e_k)=\delta_{jk}e_i$$

(Kronecker delta).

The map $T=\hat{T}|_R:R\to\mathrm{End}_K(E^{\otimes r})$ is the representation afforded by the R-module $E^{\otimes r}$.

2.6 Lemma. *For $\kappa\in R$,*

$$T(\kappa)=\sum_{i,j\in I}c_{i,j}(\kappa)e_{i,j}.$$

Proof. Let $\kappa \in R$. Since $\{e_{i,j} \mid i,j \in I\}$ is a basis for N, we have $T(\kappa) = \sum_{i,j\in I} \alpha_{i,j} e_{i,j}$ for some $\alpha_{i,j} \in K$. For each $j \in I$ we have

$$\begin{aligned}\sum_{i\in I} \alpha_{i,j} e_i &= \sum_{i,l\in I} \alpha_{i,l} e_{i,l}(e_j) \\ &= T(\kappa)(e_j) = \kappa e_j \\ &= \sum_{i\in I} c_{i,j}(\kappa) e_i \qquad \text{(Lemma 2.2)}.\end{aligned}$$

Therefore, $\alpha_{i,j} = c_{i,j}(\kappa)$ for all $i, j \in I$ and the claim follows. □

Put $S = \operatorname{im} T$.

2.7 THEOREM. $S = \operatorname{End}_{KG}(E^{\otimes r})$.

Proof. Since $R = (K\Gamma^r)^G$, it follows from Lemma 2.5 that $S = \operatorname{im} T = \operatorname{im} \hat{T}|_R \subseteq N^G = \operatorname{End}_{KG}(E^{\otimes r})$.

For the other inclusion, it is enough to show that the orthogonal complement of S in N^G is trivial, where N has the (non-degenerate) bilinear form induced by $\langle e_{i,j}, e_{k,l}\rangle = \delta_{(i,j),(k,l)}$ (Kronecker delta). Let $f = \sum_{i,j\in I} \alpha_{i,j} e_{i,j}$ be an element of this orthogonal complement. For $\kappa \in R$, we have

$$\begin{aligned}0 &= \langle f, T(\kappa)\rangle \\ &= \sum_{i,j\in I} \sum_{k,l\in I} \alpha_{i,j} c_{k,l}(\kappa) \langle e_{i,j}, e_{k,l}\rangle \qquad \text{(Lemma 2.6)} \\ &= \sum_{i,j\in I} \alpha_{i,j} c_{i,j}(\kappa).\end{aligned}$$

Since f is fixed by G, it follows that $\alpha_{i\sigma,j\sigma} = \alpha_{i,j}$ for every $\sigma \in G$. Therefore, choosing a set B of orbit representatives of $I \times I$ under the diagonal action of G, we have

$$0 = \sum_{i,j\in I} \alpha_{i,j} c_{i,j} = \sum_{(i,j)\in B} \sum_{\sigma\in G/G_{(i,j)}} \alpha_{i\sigma,j\sigma} c_{i\sigma,j\sigma} = \sum_{(i,j)\in B} |G : G_{(i,j)}| \alpha_{i,j} c_{i,j},$$

where $G_{(i,j)}$ is the stabilizer of (i,j) in G and $G/G_{(i,j)}$ is a set of representatives for the right cosets of $G_{(i,j)}$ in G, and where we have used Lemma 2.3. Using Theorem 2.4, we have $\alpha_{i,j} = 0$ for each $(i,j) \in B$ (and hence for each $i, j \in I$) so that $f = 0$ as desired. □

For a K-coalgebra C the dual space C^* has a natural structure of K-algebra [Ab, p. 55]. In particular, A^* is a K-algebra.

2.8 THEOREM. *The map $\psi : S \to A^*$ given by $\psi(T(\kappa))(c) = c(\kappa)$ is an isomorphism of K-algebras.*

Proof. By Theorem 1.3, ψ is a K-isomorphism. Using the argument of [Ma, 2.3.5 and following paragraph] with the aid of Lemma 2.6, one sees that it is an algebra homomorphism as well. □

Assume for the moment that $G = \Sigma_r$ (full symmetric group). The diagonal subgroup $D = \{(a, \dots, a) \mid a \in \Gamma\}$ of Γ^r identifies naturally with Γ. Note that $KD \subseteq R$. The functions $c_{i,j}|_{KD}$ $(i, j \in I)$ coincide with the standard basis vectors of the classical coefficient coalgebra A_r [Ma, 1.3.4]. In view of Theorem 2.4, the map $c_{i,j} \mapsto c_{i,j}|_{KD}$ defines a coalgebra isomorphism $A \cong A_r$.

Next, the image of the map $T|_{KD} : KD \to \mathrm{End}_K(E^{\otimes r})$ is the classical Schur algebra S_r [Ma, 2.1.1]. In particular, $S_r \subseteq S$. Since $S \cong A^*$ (Theorem 2.8) and $S_r \cong A_r^*$ [Ma, 2.3.5] we have $S = S_r$. (One could also see this by using Theorem 2.7 and Schur's Commutation Theorem [Ma, 2.1.3].)

In the other extreme, if $G = \{e\}$, then $S = \mathrm{End}_K(E^{\otimes r})$ (Theorem 2.7).

3. Decompositions by irreducible characters

From now on we let K be the field $\mathbf{C}$ of complex numbers and adjust the notation accordingly. Let $\mathrm{Irr}(G)$ be the set of (ordinary) irreducible characters of G and fix $\chi \in \mathrm{Irr}(G)$. Define a linear map $\tau_\chi : A \to A$ by

$$\tau_\chi(c_{i,j}) = \frac{\chi(e)}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1}) c_{i,j\sigma}.$$

3.1 Theorem. *The function τ_χ is well-defined and*

$$(\tau_\chi \otimes \tau_\chi) \circ \triangle = \triangle \circ \tau_\chi.$$

Proof. Suppose that $c_{i,j} = c_{k,l}$. By Lemma 2.3, we have $(k, l) = (i\sigma, j\sigma)$ for some $\sigma \in G$. Then

$$\begin{aligned}\sum_{\mu \in G} \chi(\mu^{-1}) c_{k,l\mu} &= \sum_{\mu \in G} \chi(\mu^{-1}) c_{i\sigma, j\sigma\mu} = \sum_{\mu \in G} \chi(\mu^{-1}) c_{i, j\sigma\mu\sigma^{-1}} \\ &= \sum_{\mu \in G} \chi(\sigma^{-1}\mu^{-1}\sigma) c_{i,j\mu} = \sum_{\mu \in G} \chi(\mu^{-1}) c_{i,j\mu},\end{aligned}$$

so $\tau_\chi(c_{k,l}) = \tau_\chi(c_{i,j})$ and τ_χ is well-defined.

Let $i, j \in I$. Using Theorem 2.4 we have

$$\begin{aligned}[(\tau_\chi \otimes \tau_\chi) \circ \triangle](c_{i,j}) &= (\tau_\chi \otimes \tau_\chi)\Big(\sum_{k \in I} c_{i,k} \otimes c_{k,j}\Big) = \sum_k \tau_\chi(c_{i,k}) \otimes \tau_\chi(c_{k,j}) \\ &= \sum_k \left(\frac{\chi(e)}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1}) c_{i,k\sigma} \right) \otimes \left(\frac{\chi(e)}{|G|} \sum_{\mu \in G} \chi(\mu^{-1}) c_{k,j\mu} \right) \\ &= \frac{\chi(e)^2}{|G|^2} \sum_{\sigma,\mu} \chi(\sigma^{-1}) \chi(\mu^{-1}) \left(\sum_k c_{i\sigma^{-1},k} \otimes c_{k,j\mu} \right) \\ &= \frac{\chi(e)^2}{|G|^2} \sum_{\sigma,\mu} \chi(\sigma^{-1}) \chi(\mu^{-1}) \triangle(c_{i\sigma^{-1},j\mu}).\end{aligned}$$

Since $c_{i\sigma^{-1},j\mu} = c_{i,j\mu\sigma}$, we can put $\rho = \mu\sigma$ on the right and get

$$[(\tau_\chi \otimes \tau_\chi) \circ \triangle](c_{i,j}) = \triangle \left[\frac{\chi(e)}{|G|} \sum_\rho \left(\frac{\chi(e)}{|G|} \sum_\sigma \chi(\sigma^{-1})\chi(\sigma\rho^{-1}) \right) c_{i,j\rho} \right]$$
$$= \triangle \left[\frac{\chi(e)}{|G|} \sum_\rho \chi(\rho^{-1}) c_{i,j\rho} \right] = [\triangle \circ \tau_\chi](c_{i,j}),$$

where we have used the generalized orthogonality relation [Is, 2.13] in the next to the last step. The theorem follows. □

Define

$$t_\chi = \frac{\chi(e)}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma \in \mathbf{C}G.$$

We have $t_\chi^2 = t_\chi$ [Is, 2.12 and proof of 2.13]. Put $E^\chi = E^{\otimes r} t_\chi$, the **symmetrized tensor space** corresponding to the character χ.

Let A_χ denote the coefficient coalgebra of the R-module E^χ and let $T_\chi : R \to \mathrm{End}_{\mathbf{C}}(E^\chi)$ be the representation afforded by E^χ. Put $S_\chi = \mathrm{im}\, T_\chi$.

The space $N = \mathrm{End}_{\mathbf{C}}(E^{\otimes r})$ is a left $\mathbf{C}G$-module with action determined by $(\sigma f)(v) = f(v\sigma)$ $(\sigma \in G)$.

3.2 Theorem.

(i) *$t_\chi S$ is a subalgebra of S and $S_\chi \cong t_\chi S$ as $\mathbf{C}$-algebras,*
(ii) *$\tau_\chi(A)$ is a subcoalgebra of A and $A_\chi = \tau_\chi(A)$.*

Proof. (i) For $f, g, \in S$ and $v \in E^{\otimes r}$, we have, using Theorem 2.7,

$$[(t_\chi f)(t_\chi g)](v) = (t_\chi f)\big(g(vt_\chi)\big) = f\big(g(vt_\chi)t_\chi\big)$$
$$= f\big(g(vt_\chi^2)\big) = f\big(g(vt_\chi)\big) = (fg)(vt_\chi)$$
$$= [t_\chi(fg)](v).$$

Therefore, $(t_\chi f)(t_\chi g) = t_\chi(fg)$, so that $t_\chi S$ is a subalgebra of S.

Define $\varphi : t_\chi S \to S_\chi$ by $\varphi(t_\chi T(\kappa)) = T_\chi(\kappa)$ $(\kappa \in R)$. We have

$$\begin{aligned} t_\chi T(\kappa) = t_\chi T(\lambda) \quad &\iff \quad \big(t_\chi T(\kappa)\big)(v) = \big(t_\chi T(\lambda)\big)(v) \quad \text{for all } v \in E^{\otimes r} \\ &\iff \quad T(\kappa)(vt_\chi) = T(\lambda)(vt_\chi) \quad \text{for all } v \in E^{\otimes r} \\ &\iff \quad T_\chi(\kappa) = T_\chi(\lambda), \end{aligned}$$

so φ is well defined and injective. It is immediate that φ is surjective and $\mathbf{C}$-linear, so it is a $\mathbf{C}$-isomorphism.

Finally, for $\kappa, \lambda \in R$, we have, using the first part of the proof,

$$\varphi((t_\chi T(\kappa))(t_\chi T(\lambda)) = \varphi(t_\chi T(\kappa)T(\lambda)) = \varphi(t_\chi T(\kappa\lambda))$$
$$= T_\chi(\kappa\lambda) = T_\chi(\kappa)T_\chi(\lambda) = \varphi(t_\chi T(\kappa))\varphi(t_\chi T(\lambda)),$$

so φ is a $\mathbf{C}$-algebra isomorphism.

(ii) First, $\tau_\chi(A)$ is a subcoalgebra of A by Theorem 3.1.

Next, we show that $\tau_\chi(A) \subseteq A_\chi$. By Lemma 1.2(ii), it is enough to show that $\tau_\chi(c_{i,j})(\kappa) = 0$ for all $\kappa \in \ker T_\chi$ and $i, j \in I$. Let $\kappa \in \ker T_\chi$ and fix $i, j \in I$.

The group G acts on the set N^R of functions from R to N from the left by $(\sigma f)(\mu) = \sigma f(\mu)$ and on N^* from the right by $(f\sigma)(e) = f(\sigma e)$.

(Step 1) $e^*_{i,j} \circ t_\chi T = e^*_{i,j} t_\chi \circ T$. We have, for $\mu \in R$,

$$(e^*_{i,j} \circ t_\chi T)(\mu) = e^*_{i,j}\big(t_\chi T(\mu)\big) = (e^*_{i,j} t_\chi)\big(T(\mu)\big) = (e^*_{i,j} t_\chi \circ T)(\mu).$$

(Step 2) $\tau_\chi(c_{i,j}) = e^*_{i,j} \circ t_\chi T$. One checks that, for $\sigma \in G$, $\sigma e_{i,j} = e_{i,j\sigma^{-1}}$ and, in turn, $e^*_{i,j}\sigma = e^*_{i,j\sigma}$. We have

$$\begin{aligned}
\tau_\chi(c_{i,j}) &= \frac{\chi(e)}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1}) c_{i,j\sigma} \\
&= \frac{\chi(e)}{|G|} \sum_{\sigma} \chi(\sigma^{-1}) e^*_{i,j\sigma} \circ T && \text{(Lemma 2.6)} \\
&= e^*_{i,j} t_\chi \circ T \\
&= e^*_{i,j} \circ t_\chi T && \text{(Step 1).}
\end{aligned}$$

(Step 3) $(t_\chi T)(\kappa) = 0$. For $v \in E^{\otimes r}$ we have

$$\big((t_\chi T)(\kappa)\big)(v) = \big(t_\chi T(\kappa)\big)(v) = T(\kappa)(v t_\chi) = \kappa(v t_\chi) = 0$$

since $v t_\chi \in E^\chi$ and $\kappa \in \ker T_\chi$.

Therefore,

$$\begin{aligned}
\tau_\chi(c_{i,j})(\kappa) &= e^*_{i,j}\big((t_\chi T)(\kappa)\big) && \text{(Step 2)} \\
&= e^*_{i,j}(0) && \text{(Step 3)} \\
&= 0,
\end{aligned}$$

and we conclude that $\tau_\chi(A) \subseteq A_\chi$.

Finally, we show that $A_\chi \subseteq \tau_\chi(A)$. For $\kappa \in R$ and $j \in I$ we have

$$\begin{aligned}
\kappa e_j t_\chi = \kappa e_j t_\chi^2 &= \frac{\chi(e)}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1}) \kappa e_{j\sigma} t_\chi \\
&= \frac{\chi(e)}{|G|} \sum_{\sigma} \chi(\sigma^{-1}) \sum_{i \in I} c_{i,j\sigma}(\kappa) e_i t_\chi && \text{(Lemma 2.2)} \\
&= \sum_i \tau_\chi(c_{i,j})(\kappa) e_i t_\chi
\end{aligned}$$

and, since some subset of $\{e_i t_\chi \mid i \in I\}$ is a basis for E^χ, the claim follows. $\square$

It follows from the orthogonality relations of characters [Is, 2.13, 2.18] that the maps τ_χ ($\chi \in \mathrm{Irr}(G)$) are pairwise orthogonal idempotents that sum to 1_A:

(i) $\tau_\chi \tau_\psi = \delta_{\chi,\psi} \tau_\chi$ (Kronecker delta),
(ii) $\sum_{\chi \in \mathrm{Irr}(G)} \tau_\chi = 1_A$.

3.3 Theorem. $A \cong \bigoplus_{\chi\in\mathrm{Irr}(G)} A_\chi$.

Proof. By (ii) of the preceding paragraph and Theorem 3.2(ii),

$$A = \sum_{\chi\in\mathrm{Irr}(G)} \tau_\chi(A) = \sum_{\chi\in\mathrm{Irr}(G)} A_\chi,$$

and by (i) of the preceding paragraph the sum is direct. □

3.4 Theorem. *The map $\psi_\chi : S_\chi \to A_\chi^*$ given by $\psi_\chi(T_\chi(\kappa))(c) = c(\kappa)$ is a* **C**-*algebra isomorphism.*

Proof. Let $\psi : S \to A^*$ be the isomorphism of Theorem 2.8 and let $\eta : S \to S_\chi$ be the epimorphism induced by restriction: $\eta(T(\kappa)) = T(\kappa)|_{E^\chi}$. Then $\psi(\ker\eta) = A_\chi^0$ (= annihilator of A_χ). Indeed, for $\kappa \in R$, we have

$$\begin{aligned}
\psi(T(\kappa)) \in A_\chi^0 &\iff \psi(T(\kappa))(c) = 0 \quad \forall\, c \in A_\chi \\
&\iff c(\kappa) = 0 \quad \forall\, c \in A_\chi \\
&\iff \kappa \in \ker T_\chi \qquad \text{(Lemma 1.2(i))} \\
&\iff T(\kappa)|_{E^\chi} = T_\chi(\kappa) = 0 \\
&\iff T(\kappa) \in \ker\eta.
\end{aligned}$$

Therefore, we have **C**-algebra isomorphisms

$$S_\chi \cong S/\ker\eta \cong A^*/A_\chi^0 \cong A_\chi^*$$

(the last **C**-isomorphism is an algebra isomorphism by Theorem 3.2(ii) and [Ab, 2.3.1(ii)]). Calling the composition ψ_χ and the composition of just the last two φ, we have

$$\psi_\chi\big(T_\chi(\kappa)\big)(c) = \varphi\big(\overline{T(\kappa)}\big)(c) = \overline{\psi\big(T(\kappa)\big)}(c) = \psi\big(T(\kappa)\big)(c) = c(\kappa).$$

□

3.5 Theorem. $S \cong \bigoplus_{\chi\in\mathrm{Irr}(G)} S_\chi$.

Proof. This follows from Theorems 2.8, 3.3, and 3.4. □

For $i \in I$, let G_i denote the stabilizer of i in G.

3.6 Theorem. *We have*

$$\dim_{\mathbf{C}} A_\chi = \frac{\chi(e)}{|G|} \sum_{(i,j)\in B} \sum_{\sigma\in G_jG_i} \chi\left(\sigma^{-1}\right),$$

where B is a set of representatives for the orbits of $I \times I$ under the diagonal action of G.

Proof. By Theorem 3.2(ii), we have $A_\chi = \tau_\chi(A)$. Since $\tau_\chi^2 = \tau_\chi$, an eigenvalue of τ_χ is either 1 or 0, so the rank of τ_χ equals its trace. Therefore, it is enough to show that the trace of τ_χ is given by the formula on the right.

Fix $i, j \in I$ and let D be a set of representatives of the (G_j, G_i) double cosets in G chosen with $e \in D$. For $\delta \in D$, let R_δ be a set of representatives of the right cosets of $G_i \cap \delta^{-1} G_j \delta$ in G_i so that

$$G_j \delta G_i = \dot{\bigcup}_{\rho \in R_\delta} G_j \delta \rho$$

(disjoint union) [Su, proof of 3.8(iv)]. We have

$$\begin{aligned} \tau_\chi(c_{i,j}) &= \frac{\chi(e)}{|G|} \sum_{\sigma \in G} \chi\left(\sigma^{-1}\right) c_{i,j\sigma} \\ &= \frac{\chi(e)}{|G|} \sum_{\delta \in D} \sum_{\substack{\mu \in G_j \\ \rho \in R_\delta}} \chi\left((\mu\delta\rho)^{-1}\right) c_{i,j(\mu\delta\rho)}. \end{aligned}$$

In the last sum, we have

$$c_{i,j(\mu\delta\rho)} = c_{i,j(\delta\rho)} = c_{i\rho^{-1},j\delta} = c_{i,j\delta}$$

so

$$\tau_\chi(c_{i,j}) = \frac{\chi(e)}{|G|} \sum_{\delta \in D} \sum_{\substack{\mu \in G_j \\ \rho \in R_\delta}} \chi\left((\mu\delta\rho)^{-1}\right) c_{i,j\delta}.$$

Let $\delta, \epsilon \in D$ and assume that $(i, j\delta) \sim (i, j\epsilon)$ so that $(i, j\delta) = (i\pi, j\epsilon\pi)$ for some $\pi \in G$. Then $\pi \in G_i$ and $\epsilon\pi\delta^{-1} \in G_j$, whence $G_j \epsilon G_i = G_j \epsilon\pi G_i = G_j \delta G_i$, implying that $\epsilon = \delta$. We conclude that the $c_{i,j\delta}$ appearing in the linear combination above are distinct and that $c_{i,j\delta} = c_{i,j}$ if and only if $\delta = e$. Therefore,

$$\begin{aligned} \operatorname{tr} \tau_\chi &= \sum_{(i,j) \in B} \left(\frac{\chi(e)}{|G|} \sum_{\substack{\mu \in G_j \\ \rho \in R_e}} \chi\big((\mu e \rho)^{-1}\big) \right) \\ &= \frac{\chi(e)}{|G|} \sum_{(i,j) \in B} \sum_{\sigma \in G_j G_i} \chi\left(\sigma^{-1}\right) \end{aligned}$$

as claimed. □

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